Addition Formulae

$$\sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\ \sin\beta$$

$$\cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\ \sin\beta$$

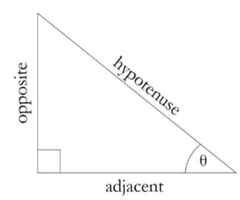
These equations are very useful for expressing the sine, cosine and tangent of multiple angles in a different format. The ≡ symbol means "identical to" (e.g. sine alpha plus beta is identical to sine alpha cosine beta plus cosine alpha sine beta). This symbols means the relationship is always true, regardless of the values of α and β.

For example, how can we find $\sin 75$?
We can split 75 into two more familiar expressions

$$\sin 75 = \sin(30 + 45) = \sin 30\cos 45 + \cos 30\sin 45$$

We should know what cos and sin of 30 and 45 are

$$\sin 75 = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$



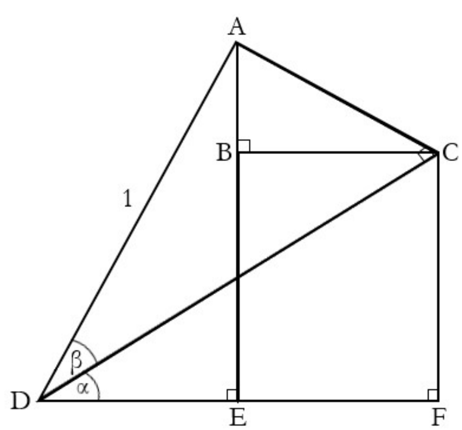
Therefore

$$\sin 75 = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

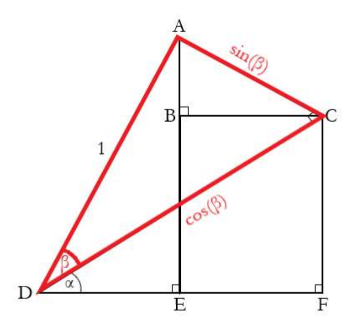
Proofs

SOH CAH TOA is used heavily in this proof. Remember that

$\sin\theta = \frac{opposite}{hypotenus}$ and $\cos\theta = \frac{adjacent}{hypotenus}$



Use the diagram on the right for the following proof. It is essentially two connected right-angled triangles so that the angle β adds to the angle α.



In triangle ACD

$$\cos\beta = \frac{DC}{1}$$

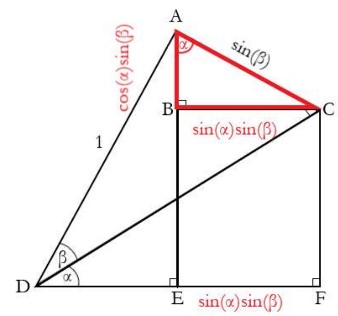
$$DC = \cos\beta$$

$$\sin\beta = \frac{AC}{1}$$

$$AC = \sin\beta$$

The angle CDF = α, and using z-angles we know that angle BCD = α and so angle ACB = 90 – α, so angle BAC = α

In triangle ABC, using the fact that $AC = \sin \beta$



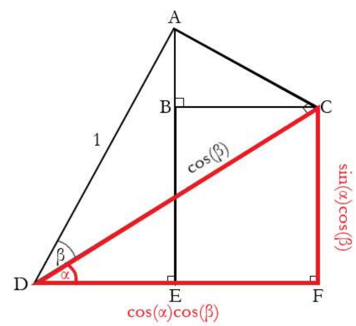
$$\cos \alpha = \frac{AB}{AC} = \frac{AB}{\sin \beta}$$

$$AB = \cos \alpha \sin \beta$$

$$\sin \alpha = \frac{BC}{AC} = \frac{BC}{\sin \beta}$$

$BC = \sin \alpha \sin \beta = EF$ (see diagram to see that $BC = EF$)

In triangle CDF, using the fact that $DC = \cos \beta$



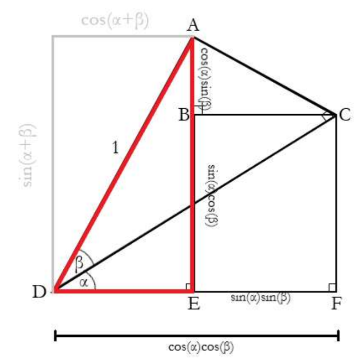
$$\cos \alpha = \frac{DF}{DC} = \frac{DF}{\cos \beta}$$

$$DF = \cos \alpha \cos \beta$$

$$\sin \alpha = \frac{CF}{\cos \beta}$$

$CF = \sin \alpha \cos \beta = BE$ (see diagram to confirm CF=BE)

In triangle ADE, using all of the information gathered



$$\cos(\alpha + \beta) = \frac{DE}{DA} = \frac{DE}{1} = DE = DF - EF$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \frac{AE}{DA} = \frac{AE}{1} = AE = BE + AB$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

There are other formulae of a similar nature. The last three are (evidently) used for subtracting angles from one another, but there is nothing fundamentally different about them.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\sin(\alpha-\beta)\equiv\sin\alpha\cos\beta-\cos\alpha\ \sin\beta$$

$$\cos(\alpha-\beta)\equiv\cos\alpha\cos\beta+\sin\alpha\ \sin\beta$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

<u>Proofs</u>

Considering the fact that $\tan\theta=\frac{\sin\theta}{\cos\theta}$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

Using the addition formulae for sine and cosine

$$\tan(\alpha+\beta)=\frac{\sin\alpha\cos\beta+\cos\alpha\ \sin\beta}{\cos\alpha\cos\beta-\sin\alpha\ \sin\beta}$$

Dividing both the top and the bottom of the fraction by $\cos\alpha$

$$\tan(\alpha+\beta)=\frac{\left(\frac{\sin\alpha\cos\beta+\cos\alpha\ \sin\beta}{\cos\alpha}\right)}{\left(\frac{\cos\alpha\cos\beta-\sin\alpha\ \sin\beta}{\cos\alpha}\right)}$$

This can be written

$$\tan(\alpha+\beta)=\frac{\left(\frac{\sin\alpha}{\cos\alpha}\cos\beta+\frac{\cos\alpha}{\cos\alpha}\sin\beta\right)}{\left(\frac{\cos\alpha}{\cos\alpha}\cos\beta-\frac{\sin\alpha}{\cos\alpha}\sin\beta\right)}$$

Cancelling and again using $\tan\theta=\frac{\sin\theta}{\cos\theta}$

$$\tan(\alpha+\beta)=\frac{\tan\alpha\cos\beta+\sin\beta}{\cos\beta-\tan\alpha\sin\beta}$$

Dividing both the top and the bottom of the fraction by $\cos\beta$

$$\tan(\alpha+\beta)=\frac{\left(\tan\alpha\frac{\cos\beta}{\cos\beta}+\frac{\sin\beta}{\cos\beta}\right)}{\left(\frac{\cos\beta}{\cos\beta}-\tan\alpha\frac{\sin\beta}{\cos\beta}\right)}$$

So

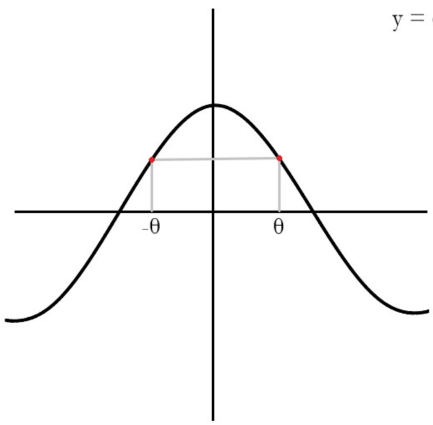
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

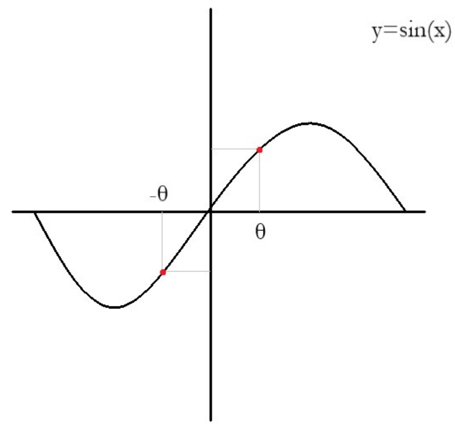
For the next formulae, consider the positive version, inputting $-\beta$ instead of β

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

Consider the graphs of $y = \cos x$ and $y = \sin x$
If $x = \theta$ or $x = -\theta$ for $y = \cos x$ it is clear they give the same value. This means $\cos x = \cos(-x)$.
If $x = \theta$ or $x = -\theta$ it is clear the y-value for one is the negative of the y-value for the other. So $\sin x = -\sin(-x)$.





Using this, we now know that

$$\sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

So

$$\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

We can use a similar argument for $\cos(\alpha-\beta)$
We know that

$$\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

So

$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

So

$$\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

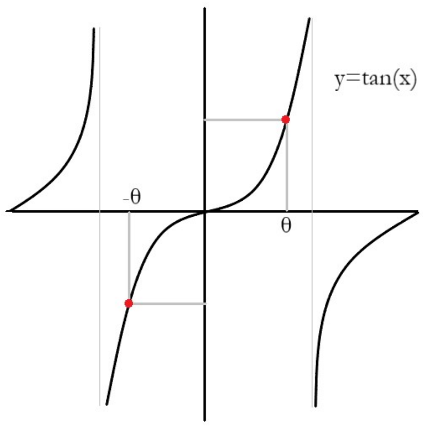
For $\tan(\alpha-\beta)$

$$\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

From the graph it is clear that $\tan x = -\tan(-x)$

so

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$



See also
- Sine, Cosine and Tangent (SOH CAH TOA)
- Double Angle Formulae

References
Attwood, G. et al. (2017). *Edexcel A level Mathematics - Pure Mathematics - Year 2*. London: Pearson Education. pp.167-168.